User friendly method to solve the inverse problem with heterogenous material properties

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Abstract. We present a novel approach to solving for the stress field and material from a measured strain field, using a partial differential equation-based approach. This approach is particularly suited for problems with heterogenous material properties. The equations have been developed in prior work, but have not been widely utilised because of the difficulty coding solvers for particular applications. Here we modify the problem formulation, such that we can exploit widely used commercial finite element solvers, dramatically increasing accessibility of the method. Furthermore, the solvers are second order accurate, in contrast to first order accuracy of previous solvers, and issues of numerical diffusion are essentially eliminated.

Introduction

Recent advances in full-field measurement techniques such as digital image correlation (DIC) have significantly enriched the data available during mechanical testing, enabling researchers to move beyond traditional test methods that rely on simplified assumptions of stress uniformity or simple geometry. These conventional tests often fall short when materials exhibit heterogeneities, localized deformations, or complex nonlinear behaviours like necking and shear banding. To address these limitations, new paradigms have emerged under the concept termed "Materials Testing 2.0" (MT2.0), where complex loading and geometries are exploited to produce heterogeneous deformation fields that can yield more comprehensive constitutive information from fewer experiments [1].

One promising approach within MT2.0 uses the known eigenvectors of measured kinematic fields (such as strain or strain rate) alongside traction boundary conditions to directly solve for unknown stress fields without explicitly assuming a specific form of constitutive model [2,3]. Unlike traditional finite element model updating (FEMU), which iteratively adjusts model parameters to match observed strains, this method directly solves a hyperbolic system of partial differential equations (PDEs) derived from equilibrium conditions. This is particularly advantageous for cases of heterogeneous material properties, as the PDEs make no assumption that the properties are uniform.

This direct inverse procedure provides stress-strain relationships at each material point with fewer underlying assumptions about material behaviour compared to other inverse identification techniques such as the virtual fields method [4]. Its non-parametric nature also makes it particularly suited for identifying local variations in mechanical properties caused by microstructural heterogeneity or manufacturing-induced variability in metallic alloys.

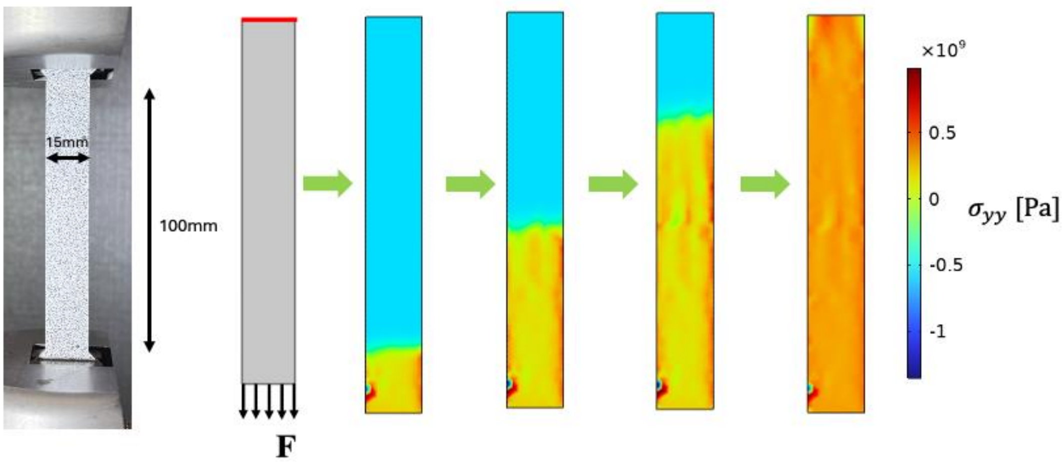

Fig 1. The new calculation procedure which artificially includes time.

A significant practical limitation of the previously developed finite volume method [5] lies in its implementation structure. The definition of the problem geometry and the application of traction boundary conditions were inherently tied to the specific problem being solved, requiring them to be hard-coded directly into the solver.

This lack of generality presents a substantial barrier for experimentalists seeking to apply the method within the Materials Testing 2.0 framework. Adapting the solver to a new specimen shape, or even modifying the boundary condition locations or types for an existing geometry, necessitates direct modification of the source code, demanding significant programming effort and familiarity with the numerical scheme, hindering wider adoption by non-experts.

This rigidity contrasts sharply with the flexibility offered by modern finite element analysis (FEA) packages. Contemporary FEA software typically allows users to define arbitrary geometries, often through importing standard CAD files or using built-in graphical modelling tools. Boundary conditions can similarly be specified on geometric entities (points, lines, surfaces) through intuitive user interfaces or high-level scripting, effectively decoupling the problem definition from the core solver engine. Furthermore, the existing finite volume implementation [5] was formulated to achieve only first-order spatial accuracy. While sufficient for initial demonstrations, first-order schemes can require significantly finer meshes to achieve high precision compared to higher-order methods, impacting computational efficiency and potentially limiting accuracy in regions with steep stress gradients.

Approach

The governing equation for the stress field, in two dimensions is:

$$\mathrm{div}(\sigma_1\, \mathbf{q}_1\, \mathbf{q}_1^T + \sigma_2\, \mathbf{q}_2\, \mathbf{q}_2^T) = 0, \tag{1}$$

where σ_1 and σ_2 are the principal stresses, and $\mathbf{q}_1$ and $\mathbf{q}_2$ are the principal directions. The fundamental assumption is that $\mathbf{q}_1$ and $\mathbf{q}_2$ are known from the strain. Hence, one has only two unknowns, σ_1 and σ_2, and two equations: force balance in the x and y direction respectively. This system is hyperbolic, but not time dependant, unlike the vast majority of hyperbolic systems.

This lack of time dependence is a major problem, as commercially developed finite element code, such as COMSOL, is written for time dependant problems. Solvers, such as the discontinuous Galerkin method written in COMSOL only accept problems in the form of:

$$\mathrm{d}\mathbf{u}/\mathrm{d}t + \mathrm{div}(\Gamma(\mathbf{u})) = 0 \tag{2}$$

Where **u** is a vector and $\boldsymbol{\Gamma}(\mathbf{u})$ is a matrix. If we choose

$$\Gamma(\mathbf{u}) = \sigma_1\, \mathbf{q}_1\, \mathbf{q}_1^T + \sigma_2\, \mathbf{q}_2\, \mathbf{q}_2^T, \tag{3}$$

then we are close to solving the problem, provided the solver gives steady state solutions. I.e. we run the finite element software, until the simulation stops changing with time. This way the $\mathrm{d}\mathbf{u}/\mathrm{d}t$ term will be zero and $\Gamma(\mathbf{u})$ solves the original equation. However, the choice of $\mathbf{u}$ is non-trivial. We find that

$$\mathbf{u} = \sigma_1\, \mathbf{q}_1 + \sigma_2\, \mathbf{q}_2 \tag{4}$$

solves this problem. This allows us to utilize COMSOL in order to solve the problem, extracting the solution when it becomes steady-state. This leads to an approach that is much easier to apply than modifying hard code, can be applied to non-trivial geometry, and is more accurate. We give a demonstration of this approach using COMSOL.

References

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